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De motu corporis ad duo centra virium fixa attracti

Leonhard Euler

DE MOTV CORPORIS AD DVO CENTRA VIRIVM FIXA ATTRACTI.

Auctore

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1.

Cum nunc quidem nullum amplius dubium superfit, quin corpora coelestia perinde moueantur, ac si se mutuo attraherent in ratione reciproca duplicata distantiarum theoria Astronomiae ad summum fastigium eueheretur, si motum quotcunque corporum se mutuo in ista ratione attrahentium definire liceret. Hinc Astronomiae perfectio a Mechanica est expectanda, ex cuius principiis cum motus illi facile ad aequationes differentiales reuocentur, totum negotium ab Analysi pendet, eaque eius parte, quae in resolutione aequationum differentialium consumitur. Quae ergo in Astronomia nondum satis sunt explorata, eorum cognitio ex sola Analysi est haurienda, cuius adhuc insignis promotio desideratur, antequam vel vnici phaenomeni perfectam explicationem reddere valeamus.

2. Quod autem in hoc negotio adhuc praestare licuit, tam exiguam vniuersae theoriae particulam complectitur, quae fere pro nihilo sit reputanda; plus enim ab Auctoribus, qui hoc argumentum tractauerunt, non est effectum, quam vt motum tantum duorum

duorum corporum, quae se mutuo in ratione reciproca distantiarum duplicata attrahunt, accurate definire docuerint. Statim ac tria corpora se hac lege attrahentia proponuntur, qui tamen casus a scopo praefixo adhuc longissime abest, cum numerus corporum in mundo se mutuo attrahentium sit maximus; omnia artificia, quae quidem adhuc in Analysi sunt inuenta, ei enodando minime sufficiunt. Et qui hoc problema sunt aggressi, plus non praestiterunt, quam vt casu, quo vnius corporis vis prae duobus reliquis valde est exigua, eorum motus vero tantum proxime assignauerint.

3. Ex hoc fonte omnia sunt hausta, quae adhuc de motu Lunae, ac de perturbationibus, quibus motus planetarum afficiuntur, sunt explorata, vbi commode vsu venit, vt vis, qua Luna ad Terram vrgetur, plurimum superet vim ad Solem directam, in planetis autem vis ad Solem tendens multo maior sit viribus, quibus in se inuicem agunt. Quae circumstantia nisi accederet, omnis opera in motuum determinatione frustra impenderetur. Quod ad Lunam attinet, cuius motum adhuc per approximationes satis exacte definire licuit, si longius a Terra esset remota, non video, quomodo vix vllam eius motus notitiam adipisci possemus, si scilicet tam longe a Terra esset remota, vt fortem satellitis Terrae esset amissura, iam in ordinem planetarum primariorum transitura. Tum nimirum eius motus mediam quandam legem sequeretur inter motum satelliti telluris et motum planetae primarii, cuius autem rationem vix vllo modo perspicere liceret, quandoquidem approximationibus nullus amplius locus relinqueretur.

4.

4. Quod si Luna Terrae esset vicinior, quam re vera est, vis perturbans a Sole profecta minueretur, ideoque Luna in motu suo circa Terram exactius leges *Keplerianas* sequeretur; aberrationes autem facilius certiusque definirentur. Quo longius autem Lunam a Terra remoueri fingamus, eo maiores aberrationes eius motum inquinabunt, quoad in eiusmodi regionem perueniat, vbi vis ad Solem tendens multum superet vim Terrae, ibique quasi Terram deserens, incipiet rationem motus planetarum primariorum sequi, verum tamen quasdam adhuc perturbationes a vi Terrae patietur, quas denuo, sed alio modo, per approximationes inuestigare licebit; perinde ac perturbationes in motibus planetarum primariorum repraesentari solent.

5. Motus autem Lunae maxime foret indeterminabilis, si quater vel quinquies a Terra magis esset remota, quam re vera est, ac si creatori libuisset, Lunae motum in tali regione assignare, Astronomi certe miris modis in eius inuestigatione, ac fortasse in cassum, defatigarentur, qui cum nunc sine auxilio Theoriae locum Lunae ad datum tempus sine notabili errore definire haud potuerint, eo casu semper in Lunae locis assignandis enormes errores essent commissuri, etiamsi fortasse innumerabiles obseruationes collegissent. Quin etiam ne suspicari quidem licet, qualem formam tum tabulis Astronomicis induci conueniret; neque patet, quomodo tabula mediorum motuum construi queat, cum eos neque ad Terram, neque ad Solem, referre liceat, multo minus intelligitur, quibusnam argumentis pro anomaliis definiendis esset vtendum. Ita tanquam eximium Astro-

 nomiae

nomiae commodum spectare debemus, quod in systemate saltem nostro solari non eiusmodi dentur corpora, de quibus dubium sit, vtrum planetis primariis an secundariis annumerari debeant.

6. In crassissima autem ignorantia circa motus coelestes versaremur, si ipsa Terra ita inter reliqua corpora fuisset disposita, vt neque legem planetarum primariorum neque secundariorum sequeretur; quoniam tum motus Solis apparens, cui cognitio reliquorum motuum innititur, nobis omnino esset inexplicabilis, quamuis plurium saeculorum obseruationes collegissemus. Vnica via ad Astronomiae notitiam perueniendi vtique per Analysin pateret, cuius beneficio problema de motu trium pluriumue corporum se mutuo attrahentium resolui deberet, neque hoc subsidio destituti quicquam in hac scientia praestare possemus. Verum etiamnum solutio huius problematis summam esset allatura vtilitatem, dum motus coelestes, quos iam proxime tantum agnoscere datur, accurate assignare valeremus; ita vt tum demum Astronomiae studium ad summum perfectionis gradum euehi sit censendum.

7. Cum igitur euolutio casus plurium corporum nequicquam ante sit expectanda, quam casus trium fuert expeditus, hic tanquam fundamentum plenioris cognitionis astronomicae spectari debet, qui propterea omnino dignus est iudicandus, ad cuius resolutionm omnes Geometrae vires suas coniunctim impendan. Maximae quidem occurrunt difficultates, quas frustra adhuc superare conati sunt Geometrae, verum fructu

inde

inde ſperandi nimis ſunt pretioſi, quam vt ab vlteriori inueſtigatione deterreri quemquam conueniat. Ac ſi hoc ipſum problema tentantes omnes vias penetrandi occluſas offendimus, quod in aliis laboribus ſaepe vſui fuit adminiculum, dum tractatio aliarum quaeſtionum affinium tandem ad quaeſitum ſcopum perduxit, eodem et hic vtamur, atque vires in aliis quaeſtionibus ſimilibus, etiamſi per ſe nullum vſum habiturae videantur, exerceamus, certa ſpe freti, inde quicquam luminis ad tenebras illas diſſipandas eſſe affulſurum.

8. Hoc igitur inſtitutum ſequens, iſtud problema tractandum ſuſcepi, vt propoſitis duobus corporibus fixis motum tertii cuiusdam corporis, quod ad vtrumque attrahatur, inueſtigarem. Sint ſcilicet corpora fixa in A et B, quorum maſſae iisdem litteris A et B indicentur, tertium autem corpus, cuius motum in eodem plano cum punctis A et B abſolui aſſumo, iam elapſo aliquo tempore t verſetur in M, cuius motus aſſignari debeat. Quod problema, etſi in mundo caſus ſimilis non occurrat, ſimilibus tamen difficultatibus, atque id cui vniuerſa Aſtronomia innititur, implicatum deprehenditur, quae autem propterea, quod hic duo corpora A et B immota finguntur, facilius ſuperari poſſe videntur; caſus enim continet aliquos per ſe perſpicuos, quorum conſideratio ad euolutionem generalem perducere videatur. Tab. III. Fig. 1.

9 Primum enim obſeruo, ſi maſſa alterius corporis A vel B euaneſcat, motum corporis M leges *Keplerianas* eſſe ſecuturum, ita vt ſectionem conicam circa

circa alterutrum punctum B vel A esset descripturum. Quod idem proxime eueniet, si corpus M ita fuerit proiectum, vt alteri corpori maneat valde vicinum, ab altero autem tantopere semper distet, vt vis eo tendens prae altera sit minima. Vnde patet motum eo magis a regulis *Keplerianis* abhorrere, quominus distantiae a punctis A et B futurae sint inaequales; hocque casu motus corporis M non adeo dissimilis videtur ei, quem secuturus esset, si corpora A et B non forent fixa, vt inde nihil luminis sperari posset. Tum vero etiam hic notari meretur casus, quo ambo corpora A et B inter se sint aequalia, corpusque M ita mouetur, vt eius orbita ad ambo aequaliter referatur; hoc enim casu motus quoque in sectione conica fieri deprehendetur.

10. His igitur obseruatis statuamus distantiam constantem $AB = a$, et variabiles $AM = v$, $BM = u$, tum vero demisso ab M in AB perpendiculo MP, sit $AP = x$, et $PM = y$, hincque $BP = a - x$ et

$$v = \sqrt{(xx + yy)} \text{ atque } u = \sqrt{((a-x)^2 + yy)}.$$

Cum iam vis acceleratrix, qua corpus M ad A attrahitur, sit $\frac{A}{vv}$, et qua ad B attrahitur vt $\frac{B}{uu}$, hinc nascetur vis secundum directionem $PA = \frac{Ax}{v^3} - \frac{B(a-x)}{u^3}$ et secundum directionem $MP = \frac{Ay}{v^3} + \frac{By}{u^3}$; ex quibus sumto elemento temporis dt constante principia mechanica has suppeditant formulas:

$$\text{I. } ddx = -2g\,dt^2\left(\frac{Ax}{v^3} - \frac{B(a-x)}{u^3}\right)$$

$$\text{II. } ddy = -2g\,dt^2\left(\frac{Ay}{v^3} + \frac{By}{u^3}\right)$$

quae

quae motus determinationem continent, vbi g eſt certa quaedam quantitas conſtans pro menſuris abſolutis introducta.

11. Cum neutra harum aequationum integrationem admittat, videndum eſt, num eas ita inter ſe combinare liceat, vt inde aequatio integrabilis exſurgat, hocque duplici modo praeſtari neceſſe eſt. Atque vna quidem huiusmodi combinatio in promtu eſt; priore enim per dx et altera per dy multiplicata ſumma prodit

$$dx\,ddx + dy\,ddy = 2g\,dt^2\left(\frac{A(x\,dx + y\,dy)}{v^3} + \frac{B(y\,dy - (a-x)\,dx)}{u^3}\right)$$

quae ob $v\,dv = x\,dx + y\,dy$ et $u\,du = y\,dy - (o-x)\,dx$ abit in hanc

$$dx\,ddx + dy\,ddy = -2g\,dt^2\left(\frac{A\,dv}{vv} + \frac{B\,du}{uu}\right)$$

cuius integralis, introducta noua conſtante, eſt

$$dx^2 + dy^2 = 4g\,dt^2\left(\frac{A}{v} + \frac{B}{u} + \frac{C}{a}\right)$$

vbi cum $\sqrt{(dx^2 + dy^2)}$ elementum curuae a corpore M tempuſculo dt deſcriptae exprimat, erit $\frac{\sqrt{(dx^2+dy^2)}}{dt}$ vera corporis M celeritas, quae ergo per diſtantias v et u commode determinatur.

12. Vna integratione expedita, vt aliam inſuper exploremus, elidamus ex formulis primo inuentis alteram maſſam, ſicque obtinebimus has aequationes:

$$(a-x)\,ddy + y\,ddx = -2g\,dt^2 . \frac{Aay}{v^3}$$

$$x\,ddy - y\,ddx = -2g\,dt^2 . \frac{Bay}{u^3},$$

vnde quidem parum lucri conſecuturi videmur. Verum ſi perpendamus eſſe

$$d.\frac{x}{v} = \frac{(xx+yy)\,dx - x(x\,dx + y\,dy)}{v^3} = \frac{y(y\,dx - x\,dy)}{v^3} \text{ et}$$

$$d.\frac{a-u}{u} = -\frac{dx((a-x)^2 + yy) - (a-x)(y\,dy - (a-x)\,dx)}{u^3} = -\frac{y(y\,dx + (a-x)\,dy}{u^3}$$

multiplicemus priorem per $x\,dy-y\,dx$ et posteriorem per $(a-x)dy+y\,dx$, habebimusque

$$(x\,dy-y\,dx)((a-x)ddy+y\,ddx)=2g\,\mathrm{A}\,a\,dt^2.d.\frac{x}{v}$$

$$((a-x)dy+y\,dx)(x\,ddy-y\,ddx)=2g\,\mathrm{B}\,a\,dt^2.d.\frac{a-x}{u}.$$

13. Cum iam sit $(a-x)ddy+y\,ddx=d.((a-x)dy+y\,dx)$ et $x\,ddy-y\,ddx=d.(x\,dy-y\,dx)$, commode euenit, vt summa harum aequationum sit integrabilis, integrali prodeunte:

$$(x\,dy-y\,dx)((a-x)dy+y\,dx)=2g\,a\,dt^2\left(\frac{\mathrm{A}x}{v}+\frac{\mathrm{B}(a-x)}{u}+\mathrm{D}\right)$$

sicque problema iam perduximus ad resolutionem aequationum differentialium primi gradus, quousque in solutione problematis de tribus corporibus mobilibus pertingere adhuc non licuit. Quodsi iam elementum temporis dt hinc elidamus, peruenimus ad hanc aequationem simpliciter differentialem:

$$a(dx^2+dy^2)\left(\frac{\mathrm{A}x}{v}+\frac{\mathrm{B}(a-x)}{u}+\mathrm{D}\right)=2(x\,dy-y\,dx)((a-x)dy+y\,dx)\left(\frac{\mathrm{A}}{v}+\frac{\mathrm{B}}{u}+\frac{\mathrm{C}}{a}\right)$$

inter binas variabiles x et y, qua natura curuae quaesitae determinatur; ita vt nunc quidem totum negotium ad resolutionem aequationis differentialis primi gradus sit perductum, quo in genere Analysis iam eximiis adminiculis est instructa.

14. Duo autem hic obstacula occurrunt, alterum quod differentialia dx et dy ad duas dimensiones ascendunt, alterum vero in quantitatibus irrationalibus v et u consistit. Quo haec obstacula facilius vincere queamus, pona-

ponamus angulos $BAM = \zeta$, $ABM = \eta$, eritque $x = v\cos.\zeta$; $y = \sin.\zeta = u\sin\eta$; et $a - x = u\cos.\eta$, vnde colligitur

$$dx^2 + dy^2 = dv^2 + vv\,d\zeta^2 = du^2 + uu\,d\eta^2$$

$$x\,dy - y\,dx = vv\,d\zeta \text{ et } (a-x)\,dy + y\,dx = uu\,d\eta$$

quibus valoribus aequatio nostra ad hanc formam simpliciorem reducitur:

$$a(dv^2 + vv\,d\zeta^2)(A\cos.\zeta + B\cos.\eta + D) = 2\,vvuu\,d\zeta\,d\eta \left(\frac{A}{v} + \frac{B}{u} + \frac{C}{a}\right).$$

Porro autem ob $v = \frac{a\sin.\eta}{\sin.(\zeta+\eta)}$ et $u = \frac{a\sin.\zeta}{\sin.(\zeta+\eta)}$ erit $x = \frac{a\cos.\zeta\sin.\eta}{\sin.(\zeta+\eta)}$ et $y = \frac{a\sin.\zeta\sin.\eta}{\sin.(\zeta+\eta)}$, hincque $dx = -\frac{a\,d\zeta\sin.\eta\cos.\eta + a\,d\eta\sin.\zeta\cos.\zeta}{\sin.(\zeta+\eta)^2}$. et $dy = \frac{a\,d\zeta\sin.\eta^2 + a\,d\eta\sin.\zeta^2}{\sin.(\zeta+\eta)^2}$, vnde fit $dx^2 + dy^2$

$$= \frac{aa(d\zeta^2\sin.\eta^2 + d\eta^2\sin.\zeta^2 - 2\,d\zeta\,d\eta\sin.\zeta\sin.\eta\cos.(\zeta+\eta))}{\sin.(\zeta+\eta)^4} = dv^2 + vv\,d\zeta^2.$$

15. Si ope horum valorum aequationem nostram ad solos binos angulos ζ et η reducamus, nanciscemur:

$$(d\zeta^2\sin.\eta^2\,d\eta^2\sin.\zeta^2 - 2\,d\zeta\,d\eta\sin\zeta\sin.\eta\cos.(\zeta+\eta))(A\cos.\zeta\ B\cos.\eta + D)$$

$$= 2\,d\zeta\,d\eta\sin.\zeta^2\sin.\eta^2\left(\frac{A\sin.(\zeta+\eta)}{\sin.\eta} + \frac{B\sin.(\zeta+\eta)}{\sin.\zeta} + C\right)$$

$$= 2\,d\zeta\,d\eta\sin.\zeta\sin.\eta(A\sin.\zeta\sin.(\zeta+\eta) + B\sin.\eta\sin(\zeta+\eta) + C\sin.\zeta\sin.\eta)$$

quae reuocatur ad hanc formam multo simpliciorem:

$$(d\zeta^2\sin.\eta^2 + d\eta^2\sin.\zeta^2)(A\cos.\zeta + B\cos.\eta + D) = 2\,d\zeta\,d\eta\sin.\zeta\sin.\eta\,(A\cos.\eta + B\cos.\zeta + C\sin.\zeta\sin.\eta + D\cos.(\zeta+\eta)).$$

Vel

Vel ob $\cos.(\zeta+\eta) = \cos.\zeta \cos.\eta - \sin.\zeta \sin.\eta$ statuamus $C - D = E$, vt habeamus:

$$d\zeta^2 \sin.\eta^2 + d\eta^2 \sin.\zeta^2 = \frac{2 d\zeta d\eta \sin.\zeta \sin.\eta (A\cos.\eta + B\cos.\zeta + D\cos.\zeta\cos.\eta + E\sin.\zeta\sin.\eta)}{A\cos.\zeta + B\cos.\eta + D}$$

vnde si ponamus breuitatis gratia

$$A\cos.\eta + B\cos.\zeta + D\cos.\zeta\cos.\eta + E\sin.\zeta\sin.\eta = P \text{ et}$$

$$A\cos.\zeta + B\cos.\eta + D = Q$$

deducimus radicem extrahendo:

$$\frac{d\zeta \sin.\eta}{d\eta \sin.\zeta} = \frac{P \pm \surd(PP - QQ)}{Q}.$$

16. Cum nulla via pateat huiusmodi aequationes resoluendi, contemplemur casus, quibus resolutio est in potestate, qui sunt, quando vel $A = 0$, vel $B = 0$; tametsi enim etiam his casibus aequatio postrema parum tractabilis videtur, tamen ex formulis principalibus solutio facile deducitur. Si enim ponamus $B = 0$, prior integratio praebet:

$$dx^2 + dy^2 = 4g\,dt\left(\frac{A}{v} + \frac{C}{a}\right)$$

tum vero ex §. 12. ob $B = 0$ impetramus

$$x\,ddy - y\,ddx = 0 \text{ hincque } x\,dy - y\,dx = \text{Const. } dt$$

ponamus ergo $(x\,dy - y\,dx)^2 = 4g\,F\,a\,dt^2$, fietque

$$Fa(dx^2 + dy^2)^2 = 2(x\,dy - y\,dx)^2\left(\frac{A}{v} + \frac{C}{a}\right) = v^4 d\zeta^2\left(\frac{A}{v} + \frac{C}{a}\right)$$

et factis substitutionibus supra indicatis:

$$F(d\zeta^2 \sin.\eta^2 + d\eta^2 \sin.\zeta^2 - 2\,d\zeta\,d\eta \sin.\zeta \sin.\eta \cos.(\zeta+\eta))$$
$$= d\zeta^2 \sin.\eta^4\left(\frac{A\sin.(\zeta+\eta)}{\sin.\eta} + C\right)$$

seu

$$d\zeta^2 \sin.\eta^2\left(1 - \frac{A\sin.\eta\sin.(\zeta+\eta) - C\sin.\eta^2}{F}\right) + d\eta^2 \sin.\zeta^2$$
$$= 2\,d\zeta\,d\eta \sin.\zeta \sin.\eta \cos.(\zeta+\eta)$$

cuius

cuius quidem resolutio vix facilior videtur, quam praecedentis; at extracta radice quadrata satis fit manifesta.

17. Verum antequam ad hanc vltimam aequationem inter ζ et η pertigimus, iam assecuti eramus aequationem duabus tantum litteris v et ζ constantem, hanc:

$$\mathrm{F}a(dv^2 + vvd\zeta^2) = v^4 d\zeta^2 \left(\frac{\mathrm{A}}{v} + \frac{\mathrm{C}}{a}\right)$$

ex qua statim elicitur

$$\mathrm{F}adv^2 = vvd\zeta^2\left(\mathrm{A}v + \frac{\mathrm{C}vv}{a} - \mathrm{F}a\right) \text{ seu}$$

$$\mathrm{F}adv^2 = v^4 d\zeta^2\left(\frac{\mathrm{C}}{a} + \frac{\mathrm{A}}{v} - \frac{\mathrm{F}a}{vv}\right)$$

vnde fit

$$\frac{dv}{vv}\sqrt{\mathrm{F}a} = d\zeta\sqrt{\left(\frac{\mathrm{C}}{a} + \frac{\mathrm{A}}{v} - \frac{\mathrm{F}a}{vv}\right)}$$

ex qua indoles sectionum conicarum more solito elici solet. Posito nempe $\frac{1}{v} = \frac{z}{a}$, fit

$$-dz = d\zeta\sqrt{\left(\frac{\mathrm{C}+\mathrm{A}z}{\mathrm{F}} - zz\right)}, \text{ vnde sequitur}$$

$$\zeta + \alpha = \text{Arc. cos. } \frac{2\mathrm{F}z - \mathrm{A}}{\sqrt{(\mathrm{AA}+4\mathrm{CF})}}$$

hincque $2\mathrm{F}z = \mathrm{A} + \cos.(\zeta+\alpha).\sqrt{(\mathrm{AA}+4\mathrm{CF})}$

ita vt sit $v = \frac{2\mathrm{F}a}{\mathrm{A}+\cos.(\zeta+\alpha).\sqrt{(\mathrm{AA}+4\mathrm{CF})}} = \frac{a\sin.\eta}{\sin.(\zeta+\eta)}$.

18. Hinc ergo aequatio integralis inter angulos ζ et η ita exprimitur, vt sit

$$\frac{2\mathrm{F}\sin.(\zeta+\eta)}{\sin.\eta} = \mathrm{A} + \cos.(\zeta+\alpha).\sqrt{(\mathrm{AA}+4\mathrm{CF})}$$

seu, quo ex dato angulo ζ angulus η facilius inueniri possit, ob $\sin.(\zeta+\eta) = \sin.\zeta\cos\eta + \cos.\zeta\sin.\eta$, erit

$$2\mathrm{F}\sin.\zeta\cot.\eta + 2\mathrm{F}\cos.\zeta = \mathrm{A} + \cos.(\zeta+\alpha).\sqrt{(\mathrm{AA}+4\mathrm{CF})}$$

seu mutata forma constantium:

$$\cot.\eta + \cot.\zeta = \frac{\mathrm{A}+\mathrm{M}\cos.\zeta+\mathrm{N}\sin.\zeta}{2\mathrm{F}\sin.\zeta}.$$

 Atque

Atque hinc simul intelligimus, si ponamus $A=0$, fore

$$\cot.\zeta+\cot.\eta=\frac{B+M'\cos.\eta+N'\sin.\eta}{2F'\sin.\eta}.$$

Quare iam constant formae, ad quas integrale aequationis differentialis §. 15. datae casibus vel $A=0$ vel $B=0$ perducatur, quo ipso via ad haec integralia perveniendi inuestigari poterit.

Pro casu $B=0$.

19. Hoc casu aequatio nostra principalis §. 15. inuenta abit in hanc formam:

$$d\zeta^2\sin.\eta^2+d\eta^2\sin.\zeta^2=\frac{2d\zeta d\eta\sin.\zeta\sin.\eta(A\cos.\eta+D\cos.\zeta\cos.\eta+E\sin.\zeta\sin.\eta)}{A\cos.\zeta+D}$$

cuius ergo nouimus integralem huiusmodi formam esse habituram:

$$\cot.\eta+\cot.\zeta=\frac{A+M\cos.\zeta+N\sin.\zeta}{2F\sin.\zeta}$$

seu breuius $\cot.\eta=\alpha+\frac{\beta+\gamma\cos.\zeta}{\sin.\zeta}$

quae igitur quomodo ex differentiali sit eruenda, investigari oportet. Quod etiamsi non difficulter per calculum statim ab initio ad hunc casum accommodatum perspiciatur, tamen consideratio corporis B calculum ita immutauit, vt haec conclusio non nisi per ambages inde colligi posse videatur. Primum autem intelligimus, loco anguli η non inutiliter eius cotangentem introductum iri; aequatione ergo per $\sin.\eta^4$ diuisa habemus

$$\frac{d\zeta^2}{\sin.\eta^2}+\sin\zeta^2(d.\cot.\eta)^2=-\frac{2d\zeta\sin.\zeta\, d.\cot.\eta(A\cot.\eta+D\cos.\zeta\cot.\eta+E\sin.\zeta)}{A\cos.\zeta+D}.$$

20. Ponamus $\cot.\eta=z$, ob $\sin\eta=\frac{1}{\surd(1+zz)}$ erit:

$$d\zeta^2(1+zz)+dz^2\sin.\zeta^2=-\frac{2d\zeta dz\sin.\zeta(z(A+D\cos.\zeta)+E\sin.\zeta)}{A\cos.\zeta+D}$$

vnde

vnde radicem extrahendo fit

$$\frac{dz\,\text{ſin.}\,\zeta}{d\zeta} = -\frac{z(A+D\,\text{coſ.}\,\zeta) - E\,\text{ſin.}\,\zeta + \sqrt{\left\{\begin{matrix}(AA-DD)zz\,\text{ſin}\,\zeta^2 + 2E(A+D\,\text{coſ.}\,\zeta)z\,\text{ſin.}\,\zeta \\ +EE\,\text{ſin}\,\zeta^2 - (A\,\text{coſ.}\,\zeta + D)^2\end{matrix}\right.}}{A\,\text{coſ.}\,\zeta + D}$$

vbi notetur, quantitatem ſigno radicali inuolutam ita referri poſſe:

$$\left(z\,\text{ſin.}\,\zeta.\sqrt{(AA-DD)} + \frac{E(A+D\,\text{coſ.}\,\zeta)}{\sqrt{(AA-DD)}}\right)^2 - \frac{(AA-DD+EE)(A\,\text{coſ.}\,\zeta+D)^2}{AA-DD}$$

Quare poſito

$$z\,\text{ſin.}\,\zeta.\sqrt{(AA-DD)} + \frac{E(A+D\,\text{coſ.}\,\zeta)}{\sqrt{(AA-DD)}} = \frac{s(A\,\text{coſ.}\,\zeta+D)\sqrt{(AA-DD+EE)}}{\sqrt{(AA-DD)}}$$

vt ſit

$$z\,\text{ſin.}\,\zeta = -\frac{E(A+D\,\text{coſ.}\,\zeta) + s(A\,\text{coſ.}\,\zeta+D)\sqrt{(AA-DD+EE)}}{AA-DD}$$

erit quantitas ſigno radicali implicata

$$\frac{(A\,\text{coſ.}\,\zeta+D)\sqrt{(AA-DD+EE)}}{\sqrt{(AA-DD)}}\sqrt{(ss-1)}.$$

21. Ponatur breuitatis gratia haec quantitas formulae irrationali aequalis $=V$, et cum noſtra aequatio ſit

$$dz\,\text{ſin}\,\zeta\,(A\,\text{coſ.}\,\zeta+D) + zd\zeta\,(A+D\,\text{coſ.}\,\zeta) + Ed\zeta\,\text{ſin}\,\zeta = Vd\zeta$$

diuidatur ea per $(A\,\text{coſ.}\,\zeta + D)^2$, et ita repraeſentari poterit

$$d.\frac{A\,z\,\text{ſin.}\,\zeta + E}{A(A\,\text{coſ.}\,\zeta+D)} = \frac{V\,d\zeta}{(A\,\text{coſ.}\,\zeta+D)^2}.$$

At per noſtram ſubſtitutionem eſt

$$A\,z\,\text{ſin.}\,\zeta + E = -\frac{DE(A\,\text{coſ.}\,\zeta+D) + As(A\,\text{coſ.}\,\zeta+D)\sqrt{(AA-DD+EE)}}{AA-DD}$$

quo valore ſubſtituto, ſimulque valore ipſius V reſtituto, erit

$$d.\frac{-(DE+As)\sqrt{(AA-DD+EE)}}{A(AA-DD)} = \frac{d\zeta\sqrt{(AA-DD+EE)}}{(A\,\text{coſ.}\,\zeta+D)\sqrt{(AA-DD)}}\sqrt{(ss-1)}$$

ſeu

$$\frac{ds}{\sqrt{(AA-DD)}} = \frac{d\zeta\sqrt{(ss-1)}}{A\,\text{coſ.}\,\zeta+D} \text{ vel } \frac{ds}{\sqrt{(ss-1)}} = \frac{d\zeta(AA-DD)}{A\,\text{coſ.}\,\zeta+D}$$

 quae

quae etiam ita repraesentari potest:

$$\frac{-ds}{\sqrt{(1-ss)}} = \frac{d\zeta\sqrt{(DD-AA)}}{A\cos.\zeta+D}$$

cuius integrale est

$$\text{Arc.}\cos.s = \text{Arc.}\cos.\frac{A+D\cos.\zeta}{A\cos.\zeta+D}+\alpha.$$

22. Cum iam sit $\text{Arc.}\cos.\frac{A+D\cos.\zeta}{A\cos.\zeta+D} = \text{Arc.}\sin.\frac{\sin.\zeta\sqrt{(DD-AA)}}{A\cos.\zeta+D}$ fiet $s = \frac{(A+D\cos.\zeta)\cos.\alpha-\sin.\alpha\sin.\zeta.\sqrt{(DD-AA)}}{A\cos.\zeta+D}$ siue hoc modo:

$$s = \frac{n(A+D\cos.\zeta)-\sin.\zeta\sqrt{(1-nn)(DD-AA)}}{A\cos.\zeta+D}.$$

vbi si fuerit $D < A$, numerum $n > 1$ capi conuenit.

Hoc itaque valore substituto aequatio integralis quaesita erit:

$$\sin.\zeta\cot.\eta = \frac{E(A+D\cos.\zeta)}{DD-AA} - \frac{\frac{n(A+D\cos.\zeta)+\sin.\zeta\sqrt{(1-nn)(DD-AA)}}{DD-AA}}{\sqrt{(AA-DD+EE)}}$$

et posito $n = \frac{E-F}{\sqrt{(AA-DD+EE)}}$ erit

$$\sin.\zeta\cot.\eta = \frac{F(A+D\cos.\zeta)}{DD-AA} + \frac{\sin.\zeta.\sqrt{(AA-DD+2EF-FF)}}{\sqrt{(DD-AA)}}$$

vbi F est quantitas constans arbitraria per nouam integrationem introducta. Ea autem mutata erit

$$\sin.\zeta\cot.\eta = \frac{A+D\cos.\zeta}{G} + \sin.\zeta.\sqrt{\left(\frac{2E}{G}+\frac{AA-DD}{GG}-1\right)}.$$

Pro casu $A=B$ et $D=E=0$.

23. Simili modo expeditur casus $A=0$, et aequatio integralis non differt a praecedente, nisi quod litterae A et B, item anguli ζ et η, inter se permutentur. Verum hoc casu, quo $A=B$, atque $D=E=0$, aequatio nostra fit

$$d\zeta^2\sin.\eta^2+d\eta^2\sin.\zeta^2 = 2d\zeta d\eta\sin.\zeta\sin.\eta$$

quae

quae manifesto praebet $d\zeta \sin.\eta = d\eta \sin.\zeta$, hincque integrando

$$l\,\mathrm{tang.}\tfrac{1}{2}\zeta = \mathrm{Const.} + l\,\mathrm{tang.}\tfrac{1}{2}\eta, \text{ vnde fit}$$

$$m\,\mathrm{tang.}\tfrac{1}{2}\zeta = n\,\mathrm{tang.}\tfrac{1}{2}\eta, \text{ seu } m(1-\cos.\zeta)\sin.\eta = n(1-\cos.\eta)\sin.\zeta$$

ita vt tangentes semissium angulorum BAM et ABM perpetuo eandem rationem seruent. Cum iam coordinatis x et y introductis sit $\cos.\zeta = \frac{x}{v}$; $\sin.\zeta = \frac{y}{v}$; $\cos.\eta = \frac{a-x}{u}$ et $\sin.\eta = \frac{y}{u}$, erit

$$\frac{m(v-x)y}{vu} = \frac{n(u-a+x)y}{vu} \text{ seu } m(v-x) = n(u-a+x)$$

ita vt sit $m(\mathrm{AM}-\mathrm{AP}) = n(\mathrm{BM}-\mathrm{BP})$.

24. Cum igitur sit $m(v-x) = n(u-a+x)$, notetur esse

$$x = \frac{aa+vv-uu}{2a} \text{ et } a-x = \frac{aa+uu-vv}{2a}, \text{ vnde fit}$$

$$m(uu-(a-v)^2) = n(vv-(a-u)^2) \text{ seu}$$

$$m(u+v-a)(u+a-v) = n(v+u-a)(v+a-u)$$

quae diuisa per $u+v-a)$ praebet

$$m(a+u-v) = n(a+v-u) \text{ seu } (m+n)(u-v) = (n-m)a$$

ita vt sit $u-v = \frac{(n-m)a}{m+n}$, quae comparetur cum hac:

$$nu-mv = na-(m+n)x, \text{ vnde colligitur}$$

$$(n-m)u = \frac{(mm+nn)a}{m+n} - (m+n)x, \text{ et}$$

$$(n-m)v = \frac{2mna}{m+n} - (m+n)x$$

quae quadrata suppeditat

$$(n-m)^2yy + (n-m)^2xx = \frac{4mmnnaa}{(m+n)^2} - 4mnax + (m+n)^2xx$$

seu $(n-m)^2yy = \frac{4mmnnaa}{(m+n)^2} - 4mnax + 4mnxx$.

 25. Su-

25. Sumamus abscissas a puncto medio C, sitque $CA = CB = b$, ideoque $a = 2b$; et ponatur $CP = z$; tum vero statuatur $m + n = b$ et $n - m = c$, et habebimus, ob $x = b - z$,

$$cv = bz - cc \quad \text{et} \quad cu = bz + cc$$

hincque $yy = \frac{bb - cc}{cc}(zz - cc)$, vnde patet, curuam esse hyperbolam centro C descriptam, cuius semiaxis $= c$, et distantia focorum a centro $CA = CB = b$, foreque tang. $\frac{1}{2}\zeta$: tang. $\frac{1}{2}\eta = b + c : b - c$. Cum porro sit $dy = \frac{zdz}{\surd(zz - cc)} \cdot \frac{\surd(bb - cc)}{c}$, erit $dx^2 + dy^2 = dy^2 + dz^2 = \frac{dz^2(bbzz - c^4)}{cc(zz - cc)}$; vnde quia ob $C = D + E = 0$ et $B = A$ habemus $dx^2 + dy^2 = 4Agdt^2\left(\frac{c}{bz - cc} + \frac{c}{bz - cc}\right) = \frac{8Abcgzdt^2}{bbzz - c^4}$, erit celeritas in $M = \frac{\surd(dx^2 + dy^2)}{dt} = \frac{2\surd 2Abcgz}{\surd(bbzz - c^4)}$? et posito $z = c$, prodit celeritas in vertice hyperbolae $= \frac{2\surd 2Abg}{\surd(bb - cc)}$. Etsi ergo hyperbola abeat in ellipsin sumendo $c > b$, tamen euidens est, motum in ellipsi absolui non posse, quia celeritas foret imaginaria, ita vt hoc casu corpus M nonnisi in hyperbola moueri possit.

26. Quemadmodum autem iste motus in hyperbola futurus sit comparatus, ex temporis ratione colligetur. Scilicet cum

sit $\surd(dx^2 + dy^2) = \frac{dz}{c}\surd\frac{bbzz - c^4}{zz - cc}$ erit

$$2dt\surd 2Abcg = \frac{dz(bbzz - c^4)}{c\surd z(zz - cc)} \text{ ideoque}$$

$$2ct\surd 2Abcg = \int\frac{dz(bbzz - c^4)}{\surd z(zz - cc)}.$$

Per reductionem autem integralium constat esse:

$$\int\frac{zzdz}{\surd z(zz - cc)} = \tfrac{2}{3}\surd z(zz - cc) + \tfrac{1}{3}cc\int\frac{dz}{\surd z(zz - cc)}$$

vnde

vnde tempus t ita determinatur, vt fit

$$2ct\sqrt{2Abcg} = \tfrac{2}{3}bb\sqrt{z(zz-cc)} + \tfrac{1}{3}cc(bb-3cc)\int\frac{dz}{\sqrt{z(zz-cc)}}.$$

Pendet ergo determinatio temporis ab integratione formulae $\int\frac{dz}{\sqrt{z(zz-cc)}}$, quam neque ad quadraturam circuli, neque hyperbolae, reduci posse constat.

27. Reducamus hanc determinationem quoque ad angulum $BAM = \zeta$, et cum sit tang $\tfrac{1}{2}\zeta = \frac{v-x}{y} = \frac{v-b+z}{y}$, habebimus tang. $\tfrac{1}{2}\zeta = \sqrt{\frac{b+c}{b-c}\cdot\frac{z-c}{z+c}}$, hincque $z = \frac{c(b-c\,cos.\zeta)}{c-b\,cos.\zeta}$, vnde fit $v = \frac{bb-cc}{c-b\,cos.\zeta}$. Quare porro nanciscimur

$\sqrt{(zz-cc)} = \frac{c\,fin.\zeta.\sqrt{(bb-cc)}}{c-b\,cos.\zeta}$ et $dz = -\frac{c(bb-cc)d\zeta\,fin.\zeta}{(c-b\,cos.\zeta)^2}$.

Ergo $\frac{dz}{\sqrt{(zz-cc)}} = -d\zeta.\sqrt{(bb-cc)}$ et

$$\frac{dz}{\sqrt{z(zz-cc)}} = -\frac{d\zeta\sqrt{(bb-cc)(c-b\,cof.\zeta)}}{\sqrt{c(b-c\,cof.\zeta)}}$$

vnde colligimus

$$2ct\sqrt{2Abcg} = \frac{2bbc\,fin.\zeta\sqrt{(bb-cc)c(b-c\,cof.\zeta)}}{3(c-b\,cof.\zeta)\sqrt{(c-b\,cof.\zeta)}}$$
$$-\tfrac{1}{3}cc(bb-3cc)\int\frac{d\zeta\sqrt{(bb-cc)(c-b\,cof.\zeta)}}{\sqrt{c(b-c\,cof.\zeta)}}$$ fiue

$$\frac{2t\sqrt{2Abg}}{\sqrt{(bb-cc)}} = \frac{2bb\,fin\,\zeta\sqrt{(b-c\,cof.\zeta)}}{3(c-b\,cof.\zeta)^{\frac{3}{2}}} - \tfrac{1}{3}(bb-3cc)\int\frac{d\zeta\sqrt{(c-b\,cof\,\zeta)}}{\sqrt{(b-c\,cof.\zeta)}}.$$

28. Casus hic notatu dignus occurrit, quo $bb = 3cc$, quoniam eo tempus algebraice assignari potest. Tum autem erit celeritas in vertice hyperbolae $= 2\sqrt{\frac{Ag\sqrt{3}}{c}} = 2\sqrt{\frac{3Ag}{b}}$. Quae celeritas si dicatur $= k$, erit

$$kt = \frac{2cc\,fin.\zeta}{c-b\,cof.\zeta}\sqrt{\frac{b-c\,cof.\zeta}{c-b\,cof.\zeta}} = \frac{2c\,fin.\zeta}{1-cof.\zeta.\sqrt{3}}\sqrt{\frac{\sqrt{3}-cof.\zeta}{1-cof.\zeta.\sqrt{3}}},$$

vel breuius ita: $\frac{t}{c}\sqrt{2Abcg} = \sqrt{z(zz-cc)}$, vnde vt ad datum tempus t definiatur locus corporis M, resolui oportet hanc aequationem cubicam:

$$z^3 - ccz = \frac{2Acgtt}{cc} = \frac{2Agtt\sqrt{3}}{c}.$$

Pro

Pro aliis autem casibus praeter tractatos vix quicquam circa motum definire licebit; hic vero occasio se obtulit eiusmodi artificia adhibendi, quae forte in vberiore huius argumenti tractatione vtilitatem afferre poterunt. Adiungamus tamen adhuc casum, quo corpus in ellipsi, cuius ambo foci sint in punctis A et B, mouebitur.

De motu corporis M in Ellipsi.

29 Quoniam casu praecedente vidimus corpus in hyperbola moueri posse, dubium est nullum, quin etiam certo quodam casu motus in ellipsi fieri queat, qui autem diuersus erit a praecedenti, quo erat $D=o$ et $E=o$, existente $A=B$. Vt autem ellipsis prodeat, necesse est, vt fiat $\text{tang.}\frac{1}{2}\zeta.\text{tang.}\frac{1}{2}\eta=m$, seu retentis valoribus $CP=z$, et $CA=CB=b$, vt fiat $(v-b+z)(u-b-z)=myy$. Cum autem sit

vel $yy=vv-(b-z)^2=(v-b+z)(v+b-z)$

vel $yy=uu-(b+z)^2=(u-b-z)(u+b+z)$

erit, vtroque seorsim adhibito,

vel $u-b-z=m(v+b-z)$, vel $v-b+z=m(u+b+z)$ quibus additis prodit $u+v-2b=m(u+v+2b)$ ita vt sit $u+v=\frac{2(1+m)b}{1-m}=2c$, seu $m=\frac{c-b}{c+b}$, denotante $2c$ axem transuersum. Cum igitur sit $uu-vv=4bz$, erit hac per illam diuisa $u-v=\frac{2bz}{c}$, ideoque $v=c-\frac{bz}{c}$ et $u=c+\frac{bz}{c}$, hincque $yy=\frac{cc-bb}{cc}(cc-zz)$.

30. Cum nunc sit $l\text{tang.}\frac{1}{2}\zeta+l\text{tang.}\frac{1}{2}\eta=lm$, erit differentiando $\frac{d\zeta}{\text{sin.}\zeta}+\frac{d\eta}{\text{sin.}\eta}=o$, hincque $\frac{d\zeta\,\text{sin.}\eta}{d\eta\,\text{sin.}\zeta}=-1$.

Quare

Quare ex §. 15. necesse est, vt sit $\frac{P-\surd(PP-QQ)}{Q}=-1$, ideoque $P+Q=0$, vnde esse oportet

$$(A+B)(\cos.\zeta+\cos.\eta)+D+D\cos.\zeta\cos.\eta+E\sin.\zeta\sin.\eta=0$$

vbi constantes ita sunt definiendae, vt haec aequatio conueniat cum natura ellipsis $\tang.\tfrac{1}{2}\zeta.\tang.\tfrac{1}{2}\eta=m=\frac{c-b}{c+b}$ seu hac $\frac{(1-\cos.\zeta)(1-\cos.\eta)}{\sin.\zeta\sin.\eta}=m=\frac{c-b}{c+b}$. Cum ergo sit $\sin.\zeta\sin.\eta=\frac{1-\cos.\zeta-\cos.\eta+\cos.\zeta\cos.\eta}{m}$, hoc valore ibi substituto fit

$$\left.\begin{array}{l} m(A+B)(\cos.\zeta+\cos.\eta)+mD+mD\cos.\zeta\cos.\eta \\ -E(\cos.\zeta+\cos.\eta)+E+E\cos.\zeta\cos.\eta \end{array}\right\}=0$$

quocirca hae conditiones requiruntur, vt sit $E=m(A+B)$ et $D=-\frac{E}{m}=-A-B$, hincque $E=\frac{c-b}{c+b}(A+B)$ et $C=D+E=\frac{-2b}{c+b}(A+B)$.

31. Vt iam motus rationem in hac ellipsi definiamus ob $dy=-\frac{z\,dz}{\surd(cc-zz)}\cdot\frac{\surd(cc-bb)}{c}$ erit $dz^2+dy^2=\frac{dz^2(c^4-bbzz)}{c\surd(cc-zz)}$ et $\surd(dz^2+dy^2)=\frac{dz\surd(c^4-bbzz)}{c\surd cc-zz)}$ supra autem inuenimus esse

$$dx^2+dy^2=4g\,dt^2\left(\frac{A}{v}+\frac{B}{u}+\frac{C}{2b}\right) \text{ seu}$$

$$dx^2+dy^2=4g\,dt^2\left(\frac{Ac}{cc-bz}+\frac{Bc}{cc+bz}-\frac{A-B}{c+b}\right)$$

quae transit in hanc formam:

$$dx^2+dy^2=\frac{4bg\,dt^2(A(c+z)(cc+bz)+B(c-z)(cc-bz))}{(c+b)(c^4-bbzz)}$$

vnde colligimus

$$\frac{4bccg\,dt^2}{b+c}=\frac{dz^2(c^4-bbzz)^2}{(cc-zz)(A(c+z)(cc+bz)+B(c-z)(cc-bz))}$$

hincque integrando

$$2ct\sqrt{\frac{bg}{b+c}}=\int\frac{(c^4-bbzz)\,dz}{\sqrt{(cc-zz)(A(c+bz)+B(c-z))(cc-bz)}}$$

32. Si ponamus $B=0$, casus reducitur ad vnicum centrum virium A, cuius calculum supra expediuimus; verum haec solutio cum illa minime conuenit, vnde methodus hic vsurpata non parum suspecta redditur. Cuius singularis phaenomeni caussam inuestigaturus obseruo, per superiorem aequationem (§. 30.) ne ambas quidem litteras D et E determinari. Ex aequatione enim $(1-\cos.\zeta)(1-\cos.\eta)=m\sin.\zeta\sin.\eta$ quadrata colligimus $(1-\cos.\zeta)(1-\cos.\eta)=mm(1+\cos.\zeta)(1+\cos.\eta)$ vnde fit

$$(1-mm)(1+\cos.\zeta\cos.\eta)=(1+mm)(\cos.\zeta+\cos.\eta).$$

Cum nunc esse debeat

$$(E+MD)(1+\cos.\zeta\cos.\eta)=(E-m(A+B))(\cos.\zeta+\cos.\eta)$$

sufficit, vt sit

$$E(1+mm)+Dm(1+mm)=E(1-mm)-(A+B)m(1-mm)$$

vnde fit $2Em+D(1+mm)+(A+B)(1-mm)=0$, seu $E=-\frac{(A+B)(1-mm)}{2m}-\frac{D(1+mm)}{2m}$, hincque

$$C=D+E=-\frac{(A+B)(1-mm)}{2m}-\frac{D(1-m)^2}{2m}.$$

33. Hinc vitium methodi, qua hic sum vsus, eo clarius in oculos incurrit. Cum enim quantitas D maneat indeterminata, etiamsi curua a corpore M descripta sit data, celeritas corporis M in quolibet orbitae suae puncto non esset determinata, sed quasi arbitrio nostro relinqueretur. Nam pro vertice ellipsis foco

A

A propiore, quo eſt diſtantia $v = c - b$, et $u = c + b$, ſeu ob $\frac{c-b}{c+b} = m$, $v = \frac{2mb}{1-m}$ et $u = \frac{2b}{1-m}$, erit celeritatis quadratum $\frac{dx^2 + dy^2}{dt^2} = \frac{2g}{b}\left(\frac{A(1-m)}{2m} + \frac{B(1-m)}{2} - \frac{(A+B)(1-mm)}{2m} - \frac{D(1-m)^2}{2m}\right) = \frac{g}{mb}(-Am(1-m) - B(1-m) - D(1-m)^2)$ ideoque ipſa celeritas $\frac{\sqrt{(dx^2+dy^2)}}{dt} = \sqrt{\frac{(1-m)g}{mb}(-Am \cdot B - D(1-m))}$, quae cum indefinita eſſe nullo modo poſſit, manifeſtum eſt, methodum §. 30. adhibitam eſſe vitioſam, id quod adhuc clarius perſpicitur, ſi ambo corpora A et B euaneſcentia ſtatuamus, quo caſu certe corpus M in linea recta eſſet inceſſurum, neque ergo ellipſim, quam hic aſſumſimus, deſcribere poterit, etiamſi calculus noſter aliter oſtendat. Plurimum igitur intererit vitium huius methodi noſſe, ne ſimili methodo alias vtentes in errores delabamur.

34. Quoniam in calculo nullum vitium deprehenditur, ipſum ratiocinium, quo vſus ſum, fallax ſit neceſſe eſt, quod iſti innititur fundamento, quod aequationi differentiali

$$\frac{d\zeta \sin.\eta}{d\eta \sin.\zeta} = \frac{P + \sqrt{(PP+QQ)}}{Q}$$ inde §. 15.

ſatisfaciat aequatio finita $(1 - \cos.\zeta)(1 - \cos.\eta) = m \sin.\zeta \sin.\eta$, (quae vtique eſt pro ellipſi) ſiquidem vna conſtantium D et E certo modo aſſumatur. Verum iam alia occaſione obſeruaui fieri poſſe, vt aequationi cuipiam differentiali aequatio quaedam finita ſatisfaciat, quae tamen in aequatione per integrationem inde deducta minime contineatur. Veluti aequationi $ds\sqrt{(1-zz)} = dz$ manifeſto ſatisfacit valor $z = 1$, qui tamen in aequatione integrali $s = \alpha + \text{Arc.} \sin. z$ vel $z = \sin.(s - \alpha)$ neuti-

neutiquam continetur, quicunque valor constanti arbitrariae α tribuatur. Nullum ergo est dubium, quin ob similem causam methodus hic adhibita in errorem induxerit.

35. Cum in originem huius erroris accuratius inquirerem, praeter expectationem incidi in completam problematis propositi solutionem, ex qua omnia, quae hactenus in hoc argumento erant desiderata, perspicue cognoscentur, simulque origo erroris hic commissi ita dilucide intelligetur, vt haud difficulter in aliis similibus casibus huiusmodi errores euitare queamus. Atque hoc modo tractatio problematis mechanici tantas dilucidationes in ipsa Analysi suppeditauit, quas alias forte frustra quaesiuissemus, quod quidem non insolitum est censendum, cum pleraque artificia, quae adhuc in Analysi sunt inuenta, quaestionibus Mechanicis ac Physicis accepta sint referenda. In his enim saepe eiusmodi inuestigationes occurrunt, quae occasionem praebent, indolem aequationum accuratius rimandi, atque adeo non raro commissio errorum illustribus inuentis fuit compensata, quemadmodum mihi hoc problema tractanti vsu venit, cuius solutionem, nisi in praedictum errorem fuissem delapsus, certe nunquam inuenissem.

Solutio completa Problematis propositi.

36. Cum problema propositum pendeat ab integratione istius aequationis differentialis:

$$\frac{d\zeta \sin.\eta}{d\eta \sin.\zeta} = \frac{P + \surd(RP - QQ)}{Q}$$

posito

posito breuitatis gratia

$$P = A\,\text{cos.}\,\eta + B\,\text{cos.}\,\zeta + D\,\text{cos.}\,\zeta\,\text{cos.}\,\eta + E\,\text{sin.}\,\zeta\,\text{sin.}\,\eta$$

$$Q = A\,\text{cos.}\,\zeta + \text{cos.}\,\eta + D$$

eam redigo ad hanc formam:

$$\frac{d\zeta\,\text{sin.}\,\eta + d\eta\,\text{sin.}\,\zeta}{d\zeta\,\text{sin.}\,\eta - d\eta\,\text{sin.}\,\zeta} = \frac{P + Q + \surd(PP - QQ)}{P - Q + \surd(PP - QQ)} = \frac{\surd(P + Q)}{\surd(P - Q)}$$

Tum vero posito tang. $\frac{1}{2}\zeta = p$, et tang. $\frac{1}{2}\eta = q$, vnde cum sit $\frac{d\zeta}{\text{sin.}\,\zeta} = \frac{dp}{p}$ et $\frac{d\eta}{\text{sin.}\,\eta} = \frac{dq}{q}$, nostra aequatio resoluenda erit

$$\frac{q\,dp + p\,dq}{q\,dp - p\,dq} = \sqrt{\frac{P + Q}{P - Q}}.$$

37. At posito tang. $\frac{1}{2}\zeta = p$ et tang. $\frac{1}{2}\eta = q$, erit $\text{sin.}\,\zeta = \frac{2p}{1 + pp}$; $\text{cos.}\,\zeta = \frac{1 - pp}{1 + pp}$; $\text{sin.}\,\eta = \frac{2q}{1 + qq}$; $\text{cos.}\,\eta = \frac{1 - qq}{1 + qq}$. Quare cum sit

$$P + Q = (A + B)(\text{cos.}\,\zeta + \text{cos.}\,\eta) + D(1 + \text{cos.}\,\zeta\,\text{cos.}\,\eta) + E\,\text{sin.}\,\zeta\,\text{sin.}\,\eta$$

fiet

$$P + Q\ \frac{2(A + B)(1 - ppqq) + 2D(1 + ppqq) + 4Epq}{(1 + pp)(1 + qq)}.$$

Deinde quia

$$P - Q = (A - B)(\text{cos.}\,\eta - \text{cos.}\,\zeta) - D(1 - \text{cos.}\,\zeta\,\text{cos.}\,\eta) + E\,\text{sin.}\,\zeta\,\text{sin.}\,\eta$$

fiet

$$P - Q = \frac{2(A - B)(pp - qq) - 2D(pp + qq) + 4Epq}{(1 + pp)(1 + qq)}.$$

His ergo valoribus introductis, nostra aequatio resoluenda erit

$$\frac{q\,dp + p\,dq}{q\,dp - p\,dq} = \sqrt{\frac{(A + B)(1 - ppqq) + D(1 + ppqq) + 2Epq}{(A - B)(pp - qq) - D(pp + qq) + 2Epq}}$$

quam facile patet ad separationem variabilium perduci posse, cum posterioris membri numerator sit functio ipsius pq, in denominatore autem quantitates p et q vbique duas dimensiones compleant.

38. Hunc in finem statuamus $pq = r$ et $\frac{p}{q} = s$ vt sit $p = \sqrt{rs}$ et $q = \sqrt{\frac{r}{s}}$, vnde ob $pdq + qdp = dr$ et $qdp - pdq = qqds$ fiet

$$\frac{dr}{qqds} = \sqrt{\frac{(A+B)(1-rr)+D(1+rr)+2Er}{qq((A+B)(ss-1)-D(ss+1)+2Es)}} \text{ seu}$$

$$\frac{dr}{ds} = \sqrt{\frac{r((A+B)(1-rr)+D(1+rr)+2Er)}{s((A-B)(ss-1)-D(ss+1)+2Es)}}$$

ex qua forma separatio variabilium r et s manifesta est, erit enim

$$\frac{dr}{\sqrt{r(A+B+D+2Er-(A+B-D)rr)}} = \frac{ds}{\sqrt{s(-A+B-D+2Es+(A-B-D)ss)}}$$

Vel si ponamus $r = xx$ et $s = yy$, habebitur

$$\frac{dx}{\sqrt{(A+B+D+2Exx-(A+B-D)x^4)}} = \frac{dy}{\sqrt{(-A+B-D+2Eyy+(A-B-D)y^4)}}$$

Quia autem r et s valores habere possent negatiuos, haec transformatio incommodum implicare posset.

39. Verum etsi hoc modo ad aequationem separatam peruenimus, tamen vtriusque partis integratio magna laborat difficultate, cum neque per circuli quadraturam, neque per logarithmos, absolui possit; constructio autem per arcus sectionum conicarum hic parum lucis esset allatura. Atque haec difficultas non minuitur, etsi statuamus B = o, quo tamen casu solutio aliunde est nota; quin etiam casus A = o et B = o, quo linea a corpore M descripta certo est recta, haud minore difficultate impeditur. Necesse igitur est, vt his casibus ambae quantitates transcendentes, quae ex vtraque integratione nascuntur, eiusmodi inter se teneant relationem, vt adeo aequationem algebraicam inter r et s complectantur. Ex quo nouus aperitur campus in aequationes algebraicas, quae forte in huiusmodi

iusmodi aequationibus differentialibus continentur, inquirendi. Atque in hoc negotio alia adhuc methodus non constat praeter eam, quam ante aliquot annos exposui, et cuius ope infinitos arcus, tam ellipticos, quam hyperbolicos, inter se comparaui, quae mihi iam tum maximum vsum aliquando habitura videbatur.

40. Sed antequam ad hanc methodum confugiam, haud abs re erit, originem erroris supra commissi indicare, quae nunc quidem est manifesta. Cum enim ad aequationem differentialem separatam inter r et s per venerimus, euidens est, ei satisfieri, si vel ipsi r eiusmodi valor constans α tribuatur, vt fiat

$$A + B + D + 2Er - (A + B - D)rr = 0$$

vel ipsi s eiusmodi valor constans β, vt fiat

$$-A + B - D + 2Es + (A - B - D)ss = 0$$

quod vtrumque duobus modis fieri potest. Atque inde quidem $r = \alpha$, sequitur $pq = \text{tang.}\tfrac{1}{2}\zeta \,\text{tang}\,\tfrac{1}{2}\eta = \alpha$, quae est aequatio pro ellipsi, hinc autem $s = \beta$ prodit $\frac{p}{q} = \beta$ seu $\text{tang.}\tfrac{1}{2}\zeta = \beta\,\text{tang.}\tfrac{1}{2}\eta$, quae est aequatio pro hyperbola. Neque vero hae curuae problema soluunt, nisi iidem valores in aequatione integrata contineantur. Euidens ergo est, nos in similem errorem illapsuros fuisse, si corporis M motum in hyperbola fieri assumsissemus.

41. Perpendamus iam aequationem integralem, quam casu $B = 0$ supra §. 22 ex nostra aequatione differentiali elicuimus, quae erit

$$\frac{\sin.\zeta \cos.\eta}{\sin.\eta} = \frac{A + D\cos.\zeta}{G} + \sin.\zeta.\, \sqrt{\left(\frac{2E}{G} + \frac{AA - DD}{GG} - 1\right)}$$

haec-

haecque, posito tang. $\frac{1}{2}\zeta = p$ et tang. $\frac{1}{2}\eta = q$, porroque $pq = r$ et $\frac{p}{q} = s$, abit in hanc formam:

$$\left.\begin{array}{l} GG(r+s)^2 + 2G(A+D)(r-s) - 8EGrs + (A+D)^2 \\ \quad + 2G(A-D)rs(r-s) \quad - 2(AA-DD)rs \\ \quad + (A-D)^2 rrss \end{array}\right\} = 0$$

quae est ergo aequatio integralis completa, huic aequationi differentiali conueniens

$$\frac{dr}{\sqrt{r(A+D+2Er-(A-D)rr)}} = \frac{ds}{\sqrt{s(-A-D+2Es+(A-D)ss)}}$$

quandoquidem in illa noua constans arbitraria G continetur.

42. Vt generaliter in talem aequationem integralem inquiramus, ponamus breuitatis gratia

$$\frac{A+B}{2E} = m; \quad \frac{A-B}{2E} = n; \quad \text{et} \quad \frac{D}{2E} = k$$

vt aequatio hoc modo integranda, siquidem id fieri potest, sit

$$\frac{dr}{\sqrt{r(m+k+r-(m-k)rr)}} = \frac{ds}{\sqrt{s(-n-k+s+(n-k)ss)}}$$

cuius integralem in hac forma contineri fingamus:

$$\mathfrak{A} + 2\mathfrak{B}r + 2\beta s + \mathfrak{C}rr + \gamma ss + 2\mathfrak{D}rs + 2\mathfrak{E}rrs + 2\varepsilon rss + \mathfrak{F}rrss = 0$$

vnde deducimus:

$$rr = \frac{-2\mathfrak{B}r - 2\mathfrak{D}rs - 2\varepsilon rss - \mathfrak{A} - 2\beta s - \gamma ss}{\mathfrak{C} + 2\mathfrak{E}s + \mathfrak{F}ss} \quad \text{et}$$

$$ss = \frac{-2\beta s - 2\mathfrak{D}rs - 2\mathfrak{E}rrs - \mathfrak{A} - 2\mathfrak{B}r - \mathfrak{C}rr}{\gamma + 2\varepsilon r + \mathfrak{F}rr}.$$

Tum vero est differentiando:

$$dr(\mathfrak{B} + \mathfrak{C}r + \mathfrak{D}s + 2\mathfrak{E}rs + \varepsilon ss + \mathfrak{F}rss) + ds(\beta + \gamma s + \mathfrak{D}r + \mathfrak{E}rr + 2\varepsilon rs + \mathfrak{F}rrs) = 0.$$

43.

43. Ex illis autem aequationibus radicem extrahendo obtinemus

$$r(\mathfrak{C}+2\mathfrak{E}s+\mathfrak{F}ss)+\mathfrak{B}+\mathfrak{D}s+\varepsilon ss=S=$$
$$\sqrt{((\mathfrak{B}+\mathfrak{D}s+\varepsilon ss)^2-(\mathfrak{A}+2\beta s+\gamma ss)(\mathfrak{C}+2\mathfrak{E}s+\mathfrak{F}ss))}$$

atque

$$s(\gamma+2\varepsilon r+\mathfrak{F}rr)+\beta+\mathfrak{D}r+\mathfrak{E}rr=-R=$$
$$-\sqrt{((\beta+\mathfrak{D}r+\mathfrak{E}rr)^2-(\mathfrak{A}+2\mathfrak{B}r+\mathfrak{C}rr)(\gamma+2\varepsilon r+\mathfrak{F}rr))}$$

qui valores in differentiali adhibito praebent

$$Sdr-Rds=0 \quad \text{seu} \quad \frac{dr}{R}=\frac{ds}{S}.$$

Supereſt ergo tantum, vt formulae irrationales R et S iis, quas noſtra aequatio reſoluenda continet, aequales efficiantur, ſeu vt fiat

$$R=\sqrt{((m+k)r+rr-(m-k)r^3)} \quad \text{et}$$
$$S=\sqrt{(-(n+k)s+ss+(n-k)s^3)}.$$

44. Cum igitur hic vtrinque tam termini primi conſtantes, quam vltimi r^4 et s^4 continentes, ſint nulli, fieri debet

$$\mathfrak{B}\mathfrak{B}-\mathfrak{A}\mathfrak{C}=0;\quad \varepsilon\varepsilon-\gamma\mathfrak{F}=0;\quad \beta\beta-\mathfrak{A}\gamma=0;\quad \mathfrak{E}\mathfrak{E}-\mathfrak{C}\mathfrak{F}=0.$$

Erit ergo

$$\mathfrak{C}=\frac{\mathfrak{B}\mathfrak{B}}{\mathfrak{A}}=\frac{\mathfrak{E}\mathfrak{E}}{\mathfrak{F}} \quad \text{et} \quad \gamma=\frac{\varepsilon\varepsilon}{\mathfrak{F}}=\frac{\beta\beta}{\mathfrak{A}}$$

hincque $\frac{\mathfrak{A}}{\mathfrak{F}}=\frac{\mathfrak{B}\mathfrak{B}}{\mathfrak{E}\mathfrak{E}}=\frac{\beta\beta}{\varepsilon\varepsilon}$.

Porro ob terminos rr et ss fieri oportet

$$\mathfrak{D}\mathfrak{D}+2\mathfrak{B}\varepsilon-\mathfrak{A}\mathfrak{F}-4\beta\mathfrak{E}-\gamma\mathfrak{C}=1$$
$$\mathfrak{D}\mathfrak{D}+2\beta\mathfrak{E}-\mathfrak{A}\mathfrak{F}-4\mathfrak{B}\varepsilon-\gamma\mathfrak{C}=1$$

 vnde

vnde fit $6\mathfrak{B}\varepsilon = 6\beta\mathfrak{E}$, seu $\frac{\mathfrak{B}}{\mathfrak{E}} = \frac{\beta}{\varepsilon}$, tum vero

$$\mathfrak{D}\mathfrak{D} - 2\beta\mathfrak{E} - \mathfrak{A}\mathfrak{F} - \gamma\mathfrak{C} = 1, \text{ seu}$$

$$\mathfrak{D}\mathfrak{D} - 2\beta\mathfrak{E} - \mathfrak{A}\mathfrak{F} - \frac{\beta\beta\mathfrak{E}\mathfrak{E}}{\mathfrak{A}\mathfrak{F}} = 1, \text{ hincque}$$

$$\mathfrak{A}\mathfrak{F}(\mathfrak{D}\mathfrak{D} - 1) = (\beta\mathfrak{E} + \mathfrak{A}\mathfrak{F})^2$$

vel $\beta\mathfrak{E} = -\mathfrak{A}\mathfrak{F} + \sqrt{\mathfrak{A}\mathfrak{F}(\mathfrak{D}\mathfrak{D} - 1)} = \mathfrak{B}\varepsilon$:

siue, cum sit $\mathfrak{F} = \frac{\mathfrak{A}\mathfrak{E}\mathfrak{E}}{\mathfrak{B}\mathfrak{B}}$, erit

$$\mathfrak{D}\mathfrak{D} = 2\beta\mathfrak{E} + \frac{\mathfrak{A}\mathfrak{A}\mathfrak{E}\mathfrak{E}}{\mathfrak{B}\mathfrak{B}} + \frac{\mathfrak{B}\mathfrak{B}\beta\beta}{\mathfrak{A}\mathfrak{A}} + 1 = \left(\frac{\mathfrak{A}\mathfrak{E}}{\mathfrak{B}} + \frac{\mathfrak{B}\beta}{\mathfrak{A}}\right)^2 + 1.$$

45. Reliqui termini dant:

$$2\beta\mathfrak{D} - 2\mathfrak{A}\varepsilon - 2\mathfrak{B}\gamma = m + k:$$
$$2\mathfrak{D}\mathfrak{E} - 2\mathfrak{C}\varepsilon - 2\mathfrak{B}\mathfrak{F} = -m + k:$$
$$2\mathfrak{B}\mathfrak{D} - 2\mathfrak{A}\mathfrak{E} - 2\beta\mathfrak{C} = -n - k:$$
$$2\mathfrak{D}\varepsilon - 2\gamma\mathfrak{E} - 2\beta\mathfrak{F} = -n - k:$$

quarum summa praebet hanc aequalitatem:

$$\mathfrak{D}(\beta + \mathfrak{B} + \varepsilon + \mathfrak{E}) - \mathfrak{A}(\varepsilon + \mathfrak{E}) - \mathfrak{F}(\beta + \mathfrak{B}) - \mathfrak{C}\varepsilon - \gamma\mathfrak{E} - \mathfrak{B}\gamma - \beta\mathfrak{C} = 0;$$

Cum nunc inuenerimus $\frac{\mathfrak{B}}{\mathfrak{E}} = \frac{\beta}{\varepsilon}$, ponamus $\mathfrak{B} = \lambda\beta$, et $\mathfrak{E} = \lambda\varepsilon$, erit $\mathfrak{C} = \frac{\lambda\lambda\beta\beta}{\mathfrak{A}}$; $\gamma = \frac{\beta\beta}{\mathfrak{A}}$; et $\mathfrak{F} = \frac{\varepsilon\varepsilon}{\beta\beta}\mathfrak{A}$; vnde ponatur porro $\mathfrak{A} = \mu\beta\beta$ et $\mathfrak{F} = \mu\varepsilon\varepsilon$; vt sit $\mathfrak{C} = \frac{\lambda\lambda}{\mu}$, et $\gamma = \frac{1}{\mu}$, hincque $\mathfrak{D}\mathfrak{D} = 1 + (\mu\beta\varepsilon + \frac{\lambda}{\mu})^2$, quibus valoribus, praeter hunc vltimum, ibi substitutis obtinebimus

$$\mathfrak{D}(\lambda + 1)(\beta + \varepsilon) - \mu\beta\varepsilon(\lambda + 1)(\beta + \varepsilon) - \frac{\lambda(\lambda + 1)(\beta + \varepsilon)}{\mu} = 0$$

seu $(\lambda + 1)(\beta + \varepsilon)(\mathfrak{D} - \mu\beta\varepsilon - \frac{\lambda}{\mu}) = 0$

cuius aequationis tres factores totidem praebent solutiones.

46. *Resolutio I.* Sit $\lambda = -1$, erit $\mathfrak{B} = -\beta$; $\mathfrak{E} = -\varepsilon$; $\mathfrak{C} = \frac{1}{\mu}$; $\gamma = \frac{1}{\mu}$; $\mathfrak{A} = \mu\beta\beta$; $\mathfrak{F} = \mu\varepsilon\varepsilon$; hincque $\mathfrak{D}\mathfrak{D} = (\mu\beta\varepsilon - \frac{1}{\mu})^2 + 1$, vnde conditiones adimplendae erunt:

$$k = \mathfrak{D}(\beta-\varepsilon) + \frac{\beta-\varepsilon}{\mu} - \mu\beta\varepsilon(\beta-\varepsilon) = (\beta-\varepsilon)(\mathfrak{D} + \tfrac{1}{\mu} - \mu\beta\varepsilon)$$

$$m = \mathfrak{D}(\beta+\varepsilon) + \frac{\beta+\varepsilon}{\mu} - \mu\beta\varepsilon(\beta+\varepsilon) = (\beta+\varepsilon)(\mathfrak{D} + \tfrac{1}{\mu} - \mu\beta\varepsilon)$$

$$n = \mathfrak{D}(\beta+\varepsilon) + \frac{\beta+\varepsilon}{\mu} - \mu\beta\varepsilon(\beta+\varepsilon) = (\beta+\varepsilon)(\mathfrak{D} + \tfrac{1}{\mu} - \mu\beta\varepsilon).$$

Hinc ergo foret $m = n$, et $B = 0$, ita vt haec resolutio tantum ad casum $B = 0$ accommodari possit. Hoc igitur casu cum sit $\frac{m}{k} = \frac{\beta+\varepsilon}{\beta-\varepsilon}$, ponatur $\beta + \varepsilon = m$ et $\beta - \varepsilon = k$, vt sit $\beta = \frac{k+m}{2}$ et $\varepsilon = \frac{m-k}{2}$; oporteteque esse $\mathfrak{D} + \frac{1}{\mu} - \mu\beta\varepsilon = 1$, vnde oritur

$$1 + 2(\mu\beta\varepsilon - \tfrac{1}{\mu}) + (\mu\beta\varepsilon - \tfrac{1}{\mu})^2 = 1 + (\mu\beta\varepsilon - \tfrac{1}{\mu})^2$$

ideoque $\mu\mu = \frac{1}{\beta\varepsilon} = \frac{4}{mm-kk}$, et $\mu = \frac{2}{\sqrt{(mm-kk)}}$. Vnde colligimus $\mathfrak{D} = 1$, eritque pro casu $m = n$ aequatio integralis

$$\mathfrak{A} + 2\mathfrak{B}r + 2\beta s + \mathfrak{C}rr + \gamma ss + 2\mathfrak{D}rs + 2\mathfrak{E}rrs + 2\varepsilon rss + \mathfrak{F}rrss = 0.$$

47. At haec aequatio integralis, quia nulla noua inest constans, non est completa; cuius ratio est, quod quantitates $\beta - \varepsilon$ et $\beta + \varepsilon$ ipsis numeris k et m non aequales, sed tantum proportionales statui debent. Sit ergo

$\beta - \varepsilon = \frac{k}{\nu}$; $\beta + \varepsilon = \frac{m}{\nu}$; erit $\beta = \frac{m+k}{2\nu}$; $\varepsilon = \frac{m-k}{2\nu}$ et $\mathfrak{D} = \nu + \mu\beta\varepsilon - \frac{1}{\mu} = \sqrt{(1 + (\mu\beta\varepsilon - \frac{1}{\mu})^2)}$, vnde fit $\mu\beta\varepsilon - \frac{1}{\mu} = \frac{1-\nu\nu}{2\nu}$ et $\mathfrak{D} = \frac{1+\nu\nu}{2\nu}$

vbi ν est quantitas arbitraria, per quam numerus μ definiri debet. Quoniam ergo β et ε per m et k cum ν dantur, aequatio integralis erit

$$0=\mu\beta\beta+2\beta(s-r)+\frac{1}{\mu}(rr+ss)+\frac{1+\nu\nu}{\nu}rs+2\varepsilon rs(s-r)$$
$$+m\varepsilon\varepsilon rrss$$

quae pro β et ε substitutis valoribus, per μ multiplicando abit in hanc formam:

$$0=\frac{\mu\mu(m+k)^2}{4\nu\nu}+\frac{\mu}{\nu}(m+k)(s-r)+rr+ss+\frac{\mu\mu}{4\nu\nu}(m-k)^2rrss$$
$$+\frac{\mu}{\nu}(m-k)rs(s-r)+\frac{\mu(1+\nu\nu)}{\nu}rs.$$

Sit $\frac{\mu}{\nu}=2f$, vt f sit constans arbitraria, sumtoque $f(mm-kk)=\frac{1}{f}+1-\nu\nu$, erit aequatio integralis completa

$$0=ff(m+k)^2+2f(m+k)(s-r)+rr+ss+ff(m-k)^2rrss$$
$$+2f(m-k)rs(s-r)+2f(1+\nu\nu)rs.$$

48. Cum vero sit $\nu\nu=1+\frac{1}{f}-f(mm-kk)$, erit $2f(1+\nu\nu)=4f+2-2ff(mm-kk)$; hincque aequatio integralis completa ita se euoluta habebit:

$$0=ff(m+k)^2+2f(m+k)(s-r)+(r+s)^2+ff(m-k)^2rrss$$
$$+2f(m-k)rs(s-r)+4frs$$
$$-2ff(mm-kk)rs$$

quae extracta radice induit hanc formam:

$$s-r+f(m+k)+f(m-k)rs=2\sqrt{rs(ff(mm-kk)-f-1)}$$

et facta restitutione $s=\frac{p}{q}$; $r=pq$ fit

$$\frac{p(1-qq)}{q}+f(m+k)+f(m-k)pp=2p\sqrt{(ff(mm-kk)-f-1}$$

quae cum integrali completo supra exhibito conuenit. Verum probe notandum, hoc integrale tantum ad casum $B=0$ pertinere.

49.

49. *Resolutio II.* Ponamus nunc $\varepsilon = -\beta$, habebimusque primo $\mathfrak{B} = \lambda\beta$; $\mathfrak{E} = -\lambda\beta$; $\mathfrak{C} = \frac{\lambda}{\mu}$; $\gamma = \frac{1}{\mu}$; $\mathfrak{A} = \mu\beta\beta$; $\mathfrak{F} = \mu\beta\beta$ et $\mathfrak{D}\mathfrak{D} = 1 + (\frac{\lambda}{\mu} - \mu\beta\beta)^2$. Tum vero hinc concludimus:

$$k = \beta(1-\lambda)(\mathfrak{D} - \tfrac{\lambda}{\mu} + \mu\beta\beta)$$

$$m = \beta(1+\lambda)(\mathfrak{D} - \tfrac{\lambda}{\mu} + \mu\beta\beta) = -n$$

ita vt haec resolutio locum non inueniat, nisi sit $m + n = 0$, ideoque $A = 0$. Pro hoc autem casu erit porro $\frac{k}{m} = \frac{1-\lambda}{1+\lambda}$, hincque $\lambda = \frac{m-k}{m+k}$, vnde sequitur $k = \frac{2\beta k}{m+k}(\mathfrak{D} - \frac{\lambda}{\mu} + \mu\beta\beta)$ seu $\mathfrak{D} = \frac{\lambda}{\mu} - \mu\beta\beta + \frac{m+k}{2\beta}$; ergo

$$1 = \frac{m+k}{\beta}(\tfrac{\lambda}{\mu} - \mu\beta\beta) + \frac{(m+k)^2}{4\beta\beta} \quad \text{et}$$

$$\frac{\lambda}{\mu} - \mu\beta\beta = \frac{\beta}{m+k} - \frac{(m+k)}{4\beta}, \text{ ideoque } \mathfrak{D} = \frac{\beta}{m+k} + \frac{m+k}{4\beta}.$$

Littera β manet indefinita, et μ definitur per hanc aequationem: $\frac{m-k}{\mu(m+k)} - \mu\beta\beta = \frac{\beta}{m+k} - \frac{(m+k)}{4\beta}$.

50. His valoribus substitutis resultat aequatio integralis completa pro casu $A = 0$:

$$0 = \mu\beta\beta + 2\lambda\beta r + 2\beta s + \frac{\lambda\lambda}{\mu}rr + \frac{1}{\mu}ss + \frac{2\beta rs}{m+k} + \frac{(m+k)}{2\beta}rs$$
$$- 2\lambda\beta rrs - 2\beta rss + \mu\beta\beta rrss$$

Statuamus $\mu\beta = f$, erit $\frac{m-k}{m+k} - ff = \frac{f}{m+k} - \frac{\mu\mu(m+k)}{4f}$ et illa aequatio per μ multiplicata erit

$$0 = ff + 2\lambda fr + 2fs + \lambda\lambda rr + ss\frac{2frs}{m+k} + \frac{\mu\mu(m+k)}{2f}rs$$
$$- 2\lambda frrs - 2frss + ffrrss$$

quae ob $\frac{\mu\mu(m-k)}{2f} = \frac{2f}{m+k} + 2ff - 2\lambda$ induit hanc formam:

$$0 = ff(1+rs)^2 + (\lambda r - s)^2 + \frac{4frs}{m+k} + 2f(\lambda r + s) - 2frs(\lambda r + s)$$

 seu

seu hanc:

$$0 = ff(1 - rs)^2 + (\lambda r + s)^2 + 2f(1 - rs)(\lambda r + s) + \frac{4frs}{m+k} + 4ffrs - 4\lambda rs$$

quae extracta radice praebet

$$f(1 - rs) + \lambda r + s = 2\sqrt{rs\left(\frac{m-k}{m+k} - ff - \frac{f}{m+k}\right)}$$

Estque haec solutio similis omnino praecedenti, dum illa ad casum $B = 0$, haec vero ad casum $A = 0$, adstringitur.

51. Tertius factor $\mathfrak{D} - \mu\beta\varepsilon - \frac{\lambda}{\mu}$ nihil monstrat, quia eius annihilatio cum aequatione $\mathfrak{D}\mathfrak{D} = 1 + (\mu\beta\varepsilon + \frac{\lambda}{\mu})^2$ consistere nequit, sicque duos tantum casus habemus, resolutionem algebraicam admittentes, scilicet si vel $B = 0$, vel $A = 0$. Praeterea vero etiam tertius casus supra euolutus, quo erat $A = B$, et $D = 0$ $E = 0$, hic sponte se offert, tum enim aequatio §. 38. abit in hanc: $\frac{dr}{\sqrt{(A+B)r(1-rr)}} = \frac{ds}{\sqrt{s.0}}$, quae subsistere nequit, nisi sit $ds = 0$, ideoque $s = \frac{p}{q} = \frac{\text{tang.}\frac{1}{2}\zeta}{\text{tang.}\frac{1}{2}\eta} = \text{Const.}$ qua aequatione hyperbola definitur. Reliquis casibus constructio aequationis

$$\frac{dr}{\sqrt{r(A+B+D+2Er-(A+B-D)rr)}} = \frac{ds}{\sqrt{s(B-A-D+2Es-(B-A+D)ss)}}$$

in subsidium est vocanda. Quod enim haec aequatio in genere integrale algebraicum non admittat, vel ex casu $D = A + B$ patet, quo prius membrum a quadratura sectionum conicarum pendet, posterius vero altiores quadraturas postulat.

52. Inuenta autem relatione inter r et s, vnde simul ratio angulorum ζ et η innotescit, cognitio motus per tempus hauritur. Cum enim sit

$$vvuud\zeta d\eta = 2gadt^2(A\cos.\zeta + B\cos.\eta + D)$$

ob

ob $v = \frac{a \sin. \eta}{\sin. (\zeta + \eta)}$ et $u = \frac{a \sin. \zeta}{\sin. (\zeta + \eta)}$, atque tang. $\frac{1}{2}\zeta = p$ et tang. $\frac{1}{2}\eta = q$, erit $d\zeta = \frac{dp \sin. \zeta}{p}$, et $d\eta = \frac{dq \sin. \eta}{q}$, seu

$$d\zeta = \frac{2dp}{1+pp} \text{ et } d\eta = \frac{2dq}{1+qq}, \text{ hincque porro}$$

$$v = \frac{aq(1+pp)}{(p+q)(1-pq)} \text{ et } u = \frac{ap(1+qq)}{(p+q)(1-pq)}, \text{ vnde fit}$$

$$\frac{4a^3 ppqq(1+pp)(1+qq)dpdq}{(p+q)^4(1-pq)^4} = 2gdt^2\left(\frac{A(1-pp)}{1+pp} + \frac{B(1-qq)}{1+qq} + D\right).$$

Fiat iam porro $pq = r$, $\frac{p}{q} = s$, seu $pp = rs$ et $qq = \frac{r}{s}$, erit $2pdp = rds + sdr$ et $2qdq = \frac{sdr - rds}{ss}$, ergo $4pqdpdq = \frac{ssdr^2 - rrds^2}{ss}$, hincque tandem:

$$\frac{a^3(r+s)(1+rs)(ssdr^2 - rrds^2)}{rs(1+s)^4(1-r)^4} = 2gdt^2\left(\frac{A(1-rs)}{1+rs} + \frac{B(s-r)}{r+s} + D\right).$$

53. Ponamus nunc breuitatis gratia:

$$RR = r(A + B + D + 2Er - (A + B - D)rr)$$
$$SS = s(B - A - D + 2Es - (B - A + D)ss)$$

vt sit $\frac{dr}{R} = \frac{ds}{S}$, statuamusque

$$\frac{dr}{R} = \frac{ds}{S} = dV, \text{ vt sit } dr = RdV \text{ et } ds = SdV, \text{ fietque}$$

$$2gdt^2 = \frac{a^3(r+s)^2(1+rs)^2(RRss - SSrr)dV^2}{rs(1+s)^4(1-r)^4(A(r+s)(1-rs) + B(s-r)(1+rs) + D(s+r)(1+rs))}.$$

Est vero

$$RRss - SSrr = rs\big(A(r+s)(1-rs) + B(s-r)(1+rs) + D(r+s)(1+rs)\big)$$

quo valore substituto prodit

$$2gdt^2 = \frac{a^3(r+s)^2(1+rs)^2 dV^2}{(1+s)^4(1-r)^4}, \text{ ideoque}$$

$$dt\sqrt{\frac{2g}{a}} = \frac{a(r+s)(1+rs)dV}{(1+s)^2(1-r)^2} = a\,dV\left(\frac{r}{(1-r)^2} + \frac{r}{(1+s)^2}\right)$$

ita vt sit

$$\frac{dt\sqrt{2g}}{a\sqrt{a}} = \frac{dr\sqrt{r}}{(1-r)^2\sqrt{(A + B + D + 2Er - (A + B - D)rr)}}$$
$$+ \frac{ds\sqrt{s}}{(1+s)^2\sqrt{(B - A - D + 2Es - (B - A + D)ss)}}$$

sicque

ſicque etiam determinatio temporis ad integrationem formularum ſimplicium eſt perducta.

54. Cum igitur hoc problema, quod primo aſpectu vix facilius quam id, quo omnia tria corpora mobilia aſſumuntur, viſum erat, perfecte reſoluere licuerit, maiorem ſpem concipimus, fore aliquando, vt et iſtud problema, cui tanquam fundamento vniuerſa Aſtronomiae inniti eſt cenſenda, reſoluatur. Equidem fateor, me hinc nullam adhuc viam ad hunc ſcopum perueniendi perſpicere, ſed etiam ad hoc plurimos ac fortaſſe operoſiſſimos conatus requiri agnoſco. Caeterum circa hoc ipſum problema, quod hic tractaui, obſeruationem Geometris forte haud ingratam adiicio, ſcilicet praeter caſus hic euolutos, innumerabiles alios dari, quibus curua, a corpore M deſcripta, futura ſit algebraica, quarum inueſtigatio Analyſi haud contemnenda incrementa allatura videtur.

55. Quanquam autem ſolutionem huius problematis ad quadraturas curuarum perduximus, tamen moleſtum foret, curuam a corpore M deſcriptam definire multoque magis ad datum tempus locum corporis aſſignare. Sin autem huiusmodi caſus in mundo exiſteret, operae pretium eſſet, hanc ſolutionem accuratius euoluere, quod hoc modo commodiſſime fieri poſſe videtur. Pro tali ſcilicet caſu, poſtquam per plura tentamina conſtantes A, B, D, E proxime ſaltem innotuerint deinceps corrigendae, tabula condi debebit pro ſingulis valoribus ipſius r valores conuenientes litterae s referens, cui deinceps tabula tempora t exhibens

bens adiungi deberet, ex qua porro viciſſim pro dato tempore t valores litterarum r et s, hincque angulos ζ et η concludere liceret. Quae determinatio ſi cum obſeruationibus minus conueniret, indicio id eſſet, conſtantes non recte eſſe aſſumtas, ſicque tandem pluribus huiusmodi tabulis conſtructis, veritas inde non difficulter erueretur.

56. Cum autem inprimis et corporis M loca noſſe conueniat, vbi eius diſtantia ab alterutro punctorum fixorum A et B eſt maxima vel minima, quemadmodum hoc definiri oporteat videamus. Quia diſtantia AM eſt $v = \frac{a\, \text{ſin.}\, \eta}{\text{ſin.}(\zeta + \eta)}$, eius differentiale

$$\frac{dv}{a} = \frac{d\eta \,\text{coſ.}\,\eta\, \text{ſin.}(\zeta+\eta) - (d\zeta + d\eta)\, \text{ſin.}\,\eta\, \text{coſ.}(\zeta+\eta)}{\text{ſin.}(\zeta+\eta)^2} = \frac{d\eta\, \text{ſin.}\,\zeta - d\zeta\, \text{ſin.}\,\eta\, \text{coſ.}(\zeta+\eta)}{\text{ſin.}(\zeta+\eta)^2}$$

nihilo aequale poſitum, ſeu $\frac{d\eta}{\text{ſin.}\,\eta} = \frac{d\zeta}{\text{ſin.}\,\zeta} \text{coſ.}(\zeta+\eta)$ indicabit loca, quibus diſtantia AM eſt vel maxima, vel minima. Poſito ergo tang. $\frac{1}{2}\zeta = p$ et tang. $\frac{1}{2}\eta = q$, ob coſ. $(\zeta+\eta) = \frac{(1-pp)(1-qq)-4pq}{(1+pp)(1+qq)}$, habebimus $\frac{dq}{q} = \frac{dp}{p} \cdot \frac{(1-pq)^2-(p+q)^2}{(1+pp)(1+qq)}$. Factoque porro $pq = r$, $\frac{p}{q} = s$, ſeu $p = \sqrt{rs}$, $q = \sqrt{\frac{r}{s}}$, fiet $(\frac{dr}{r} - \frac{ds}{s})(s(1+rr) + r(1+ss)) = (\frac{dr}{r} + \frac{ds}{s})(s(1-r)^2 - r(1+s)^2)$ ſeu $dr(1+s)^2 = ds(1-r)^2$. Vbi ergo eſt $\frac{dr}{(1-r)^2} = \frac{ds}{(1+s)^2}$ ibi diſtantia AM $= v$ eſt vel maxima vel minima.

57. Cum igitur ſupra inuenerimus $\frac{dr}{R} = \frac{ds}{S}$, pro his locis habemus $\frac{R}{(1-r)^2} = \frac{S}{(1+s)^2}$, vnde relatio inter quantitates finitas r et s eruitur, quae eſt:

$$r(1+s)^4(A+B+D+2Er-(A+B-D)rr) =$$
$$s(1-r)^4(B-A-D+2Es-(B-A+D)ss)$$

 vnde

vnde posito breuitatis gratia

$$\frac{A+B+D}{2E}=m,\quad \frac{B-A-D}{2E}=n$$
$$\frac{A+B-D}{2E}=\mu,\quad \frac{B-A+D}{2E}=\nu$$

vt sit $\mu+\nu=m+n$, enascitur haec aequatio

$$\left.\begin{array}{lllll} +mr+rr & -\mu r^3 & +4(n-\mu)r^4s-nr^4s & & -r^4ss \\ -ns+4(m+n)rs & +(4-6n)rrs & +4(m-\nu)rs^3 & +(4-6\mu)r^3ss-4(\mu+\nu)r^3s^3 & +\nu r^4s^3 \\ -ss & +(4+6m)rss & & +(4+6r)rrs^3+rrs^4 & +\mu r^3s^4 \\ & +\nu s^3 & & +mrs^4 & \end{array}\right\}=0$$

quae aequatio in genere nullos factores habere videtur. At aequatio inter p et q erit

$$(p+q)^4(m+pq-\mu ppqq)=(1-pq)^4(nqq+pq-\nu pp).$$

58. Restitutis autem ipsis angulis ζ et η, inter eos aequatio pro hoc casu, quo distantia v fit vel maxima vel minima, ita se habebit:

$$\sin.\left(\tfrac{\zeta+\eta}{2}\right)^4(D(1+\cos.\zeta\cos.\eta)+(A+B)(\cos.\zeta+\cos.\eta)$$
$$+E\sin.\zeta\sin.\eta)=$$
$$\cos.\left(\tfrac{\zeta+\eta}{2}\right)^4(D(\cos.\zeta\cos.\eta-1)+(B-A)(\cos.\zeta-\cos.\eta)$$
$$+E\sin.\zeta\sin.\eta)$$

vbi est $\sin\left(\tfrac{\zeta+\eta}{2}\right)^4=\tfrac{1}{4}(1-\cos(\zeta+\eta))^2$ et $\cos.\left(\tfrac{\zeta+\eta}{2}\right)^4=\tfrac{1}{4}(1+\cos.(\zeta+\eta))^2$ vnde colligimus:

$$(1+\cos.(\zeta+\eta)^2)(A\cos.\zeta+B\cos.\eta+D)=2\cos.(\zeta+\eta)$$
$$(A\cos.\eta+B\cos.\zeta+D\cos.\zeta\cos.\eta+E\sin.\zeta\sin.\eta)$$

quae autem aequatio aeque parum resolutionem admittit. Caeterum quia permutatis angulis ζ, η et massis A et B aequatio non mutatur, eadem loca indicat, vbi distantia $BM=u$ fit vel maxima vel minima. Verum radice quadrata extracta bini casus a se inuicem separantur.

DE

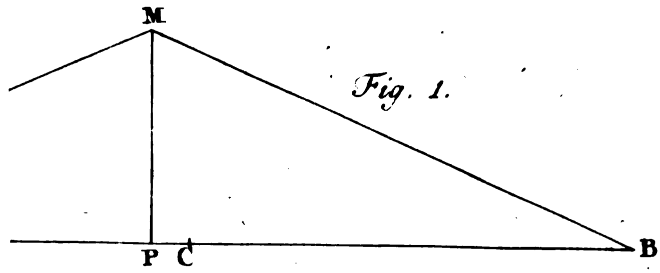
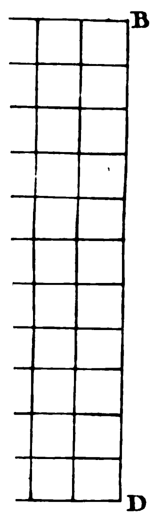


Fig. 1.



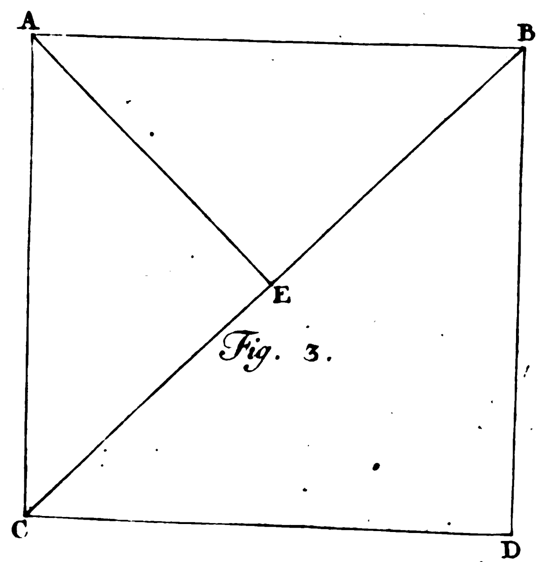


Fig. 3.

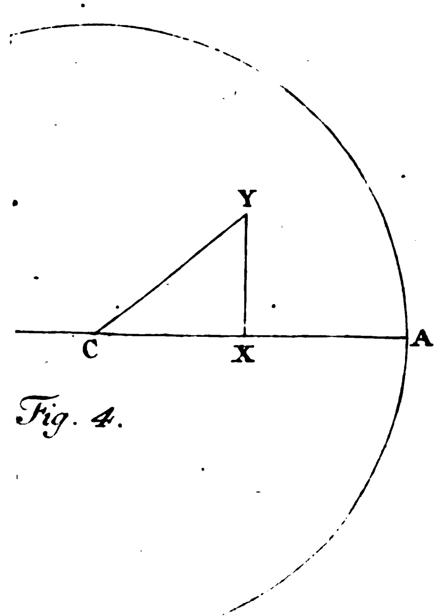


Fig. 4.